deviation of the temperature excess $\Theta_1^*(0, 0, 0, Fo)$ for the three-dimensional case from the corresponding quantity in the one-dimensional case; q_0, constant heat flux density inside the prescribed square region on the surface of the body; T_0, initial temperature of the body; $2l$, length of a side of the square heater; τ_1, time corresponding to Fo_1.

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SANDWICH PLATE UNDER THERMAL IMPACT

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An expression is obtained for the temperature field and the fluctuations excited by a thermal impact are investigated.

The extensive application of laminated structure elements in industry arouses interest in determining the temperature fields therein and in describing their dynamic behavior under thermal force action. The vibrations of a circular single-plate excited by a thermal impact are considered in the monograph [1]. Similar investigations are performed in this paper for sandwich circular plates of nonsymmetrical thickness, assembled from materials with different thermophysical and mechanical properties.

Let us consider an unlimited sandwich plate of thickness $h = \sum_{k=1}^{3} h_k$ (k = 1, 2, 3; h_1, h_2 are the thicknesses of the outer layers and $h_3 = 2c$ is the filler thickness), on whose outer surface of the heat shielding layer 1 ($z = c + h_1$) a thermal flux of density q_t acts in a direction normal to the surface. The outer plane of the carrying layer 2 ($z = -c - h_2$) is assumed heat-insulated. A cylindrical r, φ, z coordinate system is coupled to the filler middle surface, and the z axis is directed toward the layer 1. Under the mentioned heat-transfer conditions the temperature field in the k-th layer of the plate $\theta_k(z, t) = T - T_0$ (T_0 is the initial temperature) satisfies the heat conduction equation

$$\theta_{k,zz} = \dot{\theta}_k / a_{kt}, \tag{1}$$

under the initial (t = 0, t is the time)

$$\theta_k(z, 0) = 0 \tag{2}$$

and boundary

$$\left.\begin{array}{l} \lambda_1\theta_{1,z} = -q_t \ (z = c + h_1),\ \theta_1 = \theta_3,\ \lambda_1\theta_{1,z} = \lambda_3\theta_{3,z} \ (z = c), \\ \theta_2 = \theta_3,\ \lambda_2\theta_{2,z} = \lambda_3\theta_{3,z} \ (z = -c),\ \theta_{2,z} = 0 \ (z = -c - h_2) \end{array}\right\} \tag{3}$$

conditions. Here $a_{kt} = \lambda_k/(c_{kt}\rho_k)$ is the thermal diffucivity of the k-th layer, the comma in the subscript denotes the operation of differentiation with respect to the subsequent coordinates.

The solution of the boundary-value problem (1) under the initial (2) and boundary (3) condition is executed by an operational method based on the Laplace transform [2].

Analysis of the analytical expressions obtained for the temperature fields in each of the layers and comparing them with known solutions (for $h_2 = 0$ the field for a bilayer plate presented in [2] follows, while for $h_1 \ll h_3$ we obtain the temperature field of a thin coat-

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ing [3]), and the results of the numerical computation on an electronic computer permit representation of the temperature fields in the system under study in the following form with accuracy (5-7%) sufficient for engineering practice for our specific conditions ($h_1 \approx h_2 \approx h_3$)

$$\theta_1(z,\ t) = \frac{q_t \sqrt{a_{1t}t}}{\lambda_1}\left\{\lambda_{13}^{+}\left[\operatorname{ierfc} D_1(x) + \operatorname{ierfc} D_1(-x) - \frac{2}{\sqrt{\pi}} - \right.\right.$$

$$\left. - \frac{h_1}{\sqrt{a_{1t}t}} - \frac{2h_1^2}{3\sqrt{\pi}\, a_{1t}t}\right] + \lambda_{13}^{-}\left[\operatorname{ierfc} D_2(x) + \operatorname{ierfc} D_2(-x) - \right.$$

$$\left.\left. - \operatorname{ierfc} D_3(x) - \operatorname{ierfc} D_3(-x)\right] + 2\operatorname{ierfc}\frac{x}{2\sqrt{a_{1t}t}}\right\},$$

$$\theta_2(z,\ t) = \frac{4q_t\sqrt{a_{1t}t}}{\lambda_1\lambda_{23}^{+}}\left\{\operatorname{ierfc}\frac{h_3-(x-2h_2)\sqrt{a_{3t}/a_{2t}}}{2\sqrt{a_{3t}t}} + \right.$$

$$\left. + \operatorname{ierfc}\frac{3h_3-(x-2h_2)\sqrt{a_{3t}/a_{2t}}}{2\sqrt{a_{3t}t}} + \operatorname{ierfc}\frac{h_3+x\sqrt{a_{3t}/a_{2t}}}{2\sqrt{a_{3t}t}} + \operatorname{ierfc}\frac{3h_3+x\sqrt{a_{3t}/a_{2t}}}{2\sqrt{a_{3t}t}}\right\},$$

$$\theta_3(z,\ t) = \frac{2q_t\sqrt{a_{1t}t}}{\lambda_1}\left\{\operatorname{ierfc}\frac{4h_3-x}{2\sqrt{a_{3t}t}} + \operatorname{ierfc}\frac{6h_3+x}{2\sqrt{a_{3t}t}} - \right.$$

$$\left. - \operatorname{ierfc}\frac{2h_3+x}{2\sqrt{a_{3t}t}} - \frac{1}{\sqrt{\pi}} - \frac{x}{4\sqrt{a_{3t}t}} - \frac{x^2}{12\sqrt{\pi}\, a_{3t}t}\right\}. \tag{4}$$

Here

$$\lambda_{13}^{+} = 1 + \frac{\lambda_3}{\lambda_1}\sqrt{\frac{a_{1t}}{a_{3t}}};\ \lambda_{13}^{-} = 1 - \frac{\lambda_3}{\lambda_1}\sqrt{\frac{a_{1t}}{a_{3t}}};$$

$$\lambda_{23}^{+} = 1 + \frac{\lambda_2}{\lambda_3}\sqrt{\frac{a_{3t}}{a_{2t}}};\ x = -z + h_1 + c;$$

$$D_1(\pm x) = \frac{4h_3\sqrt{a_{1t}/a_{3t}} - (2h_1 \pm x)}{2\sqrt{a_{1t}t}};\ D_2(\pm x) =$$

$$= \frac{6h_3\sqrt{a_{1t}/a_{3t}} + 2h_1 \pm x}{2\sqrt{a_{1t}t}};\ D_3(\pm x) = \frac{2h_3\sqrt{a_{1t}/a_{3t}} + 2h_1 \pm x}{2\sqrt{a_{1t}t}};$$

and ierfc y are functions known in the theory of heat conduction [2, 3].

Thermal impact on the surface of a sandwich plate can cause it to vibrate. For a circular plate we introduce an assumption about the heat insulation of its outline, which permits description of the temperature field therein by using (4).

We take geometric hypotheses in conformity with the É. I. Grigolyuk model [4]: the Kirchhoff assumptions are valid in the carrying layers, while the deformed normal remains rectilinear and incompressible in the stiff filler. In this case the radial displacements in the layers u_r^k can be expressed in terms of three unknown functions: u(r, t) is the radial displacement of the coordinate plane, ψ(r, t) is shear in the filler, and w(r, t) is deflection of the plate

$$u_r^{(1)} = u + c\psi - zw_{,r};\ u_r^{(2)} = u - c\psi - zw_{,r};\ u_r^{(3)} = u + z\psi - zw_{,r}. \tag{5}$$

There are no tangential displacements because of the axial symmetry of the problem.

The deformation components are determined from the known formulas

$$\varepsilon_r^k = u_{r,r}^k;\ \varepsilon_\varphi^k = u_r^k/r;\ 2\varepsilon_{rz}^{(3)} = \psi. \tag{6}$$

The stress connection to the strain is described by thermoelasticity relations

$$s_\alpha^k = 2G_k\, э_\alpha^k;\ \sigma^k = 3K_k(\varepsilon^k - \alpha_k\theta^k);\ s_{rz}^{(3)} = G_3\psi\ (\alpha = r,\ \varphi). \tag{7}$$

Here α_k is the coefficient of linear temperature elongation.

The theorem of the minimum of the potential strain energy is used in deriving the equation of plate motion. Its variation has the form

$$\delta\Pi = \iint \Big(\sum_{\alpha, k} \int_{h_k} \sigma_\alpha^k \, \delta\varepsilon_\alpha^k \, dz + 2 \int_{h_3} \sigma_{rz}^{(3)} \delta\varepsilon_{rz}^{(3)} dz \Big) r dr d\varphi. \tag{8}$$

Later, as in [5], the relative smallness of the filler shear modulus $G_3 \ll G_1, G_2$ is used, whereupon the work of the tangential stress $\sigma_{rz}^{(3)}$ in (8) can be neglected, which equals $2cG_3\psi$. The general form of the equations of motion in generalized internal forces and moments agrees with the traditional form for laminar plates [5, 6].

Use of the relationship (5)-(7) permits reducing the equation of motion in the form

$$\begin{gathered} L_2(a_1 u + a_2\psi - a_3 w_{,r}) = 0; \;\; L_2(a_2 u + a_4\psi - a_5 w_{,r}) = 0; \\ L_3(a_3 u + a_5\psi - a_6 w_{,r}) - M_0 r_0^2 \ddot{w} = 0. \end{gathered} \tag{9}$$

The initial motion conditions are

$$w(r, 0) = 0; \;\; \dot{w}(r, 0) = 0; \;\; \theta(z, 0) = 0. \tag{10}$$

Hinge-support over the plate boundary contour with the presence of a stiff diaphragm preventing the relative shear of the layers at the endplate is taken as boundary conditions

$$M_r \equiv a_2 u_{,r} + a_5 \psi_{,r} - a_6 w_{,rr} - a_{60} w_{,r}/r - M_t = 0; \;\; u = \psi = w = 0 \;\; (r = 1). \tag{11}$$

Here and henceforth, the geometric parameters and linear coordinates are referred to the plate radius r_0

$$\begin{gathered} L_2(u) \equiv ((ru)_{,r}/r)_{,r} \equiv u_{,rr} + u_{,r}/r - u/r^2; \\ L_3(u) \equiv (rL_2(u))_{,r}/r \equiv u_{,rrr} + 2u_{,rr}/r - u_{,r}/r^2; \\ a_1 = \sum_{k=1}^{3} h_k K_{k0}; \;\; a_2 = c(h_1 K_{10} + h_2 K_{20}); \;\; a_3 = h_1(c + h_1/2) K_{10} - \\ - h_2(c + h_2/2) K_{20}; \;\; a_4 = c^2(h_1 K_{10} + h_2 K_{20} + h_3 K_{30}/3); \\ a_5 = c(h_1(c + h_1/2) K_{10} + h_2(c + h_2/2) K_{20} + 2c^2 K_{30}/3); \\ a_6 = h_1(c^2 + ch_1 + h_1^2/3) K_{10} + h_2(c^2 + ch_2 + h_2^2/3) K_{20} + 2c^3 K_{30}/3; \\ a_{60} = a_6\{(K_k - 2/3G_k) \to K_{k0}\}; \;\; K_{k0} = K_k + 4/3G_k; \\ M_t = 3\sum_{k=1}^{3} \alpha_k (K_k - 2/3G_k) \int_{h_k} \theta_k z dz; \;\; M_0 = \sum_{k=1}^{3} \rho_k h_k, \end{gathered}$$

where ρ_k is the density of the material of the k-th layer.

After integration by parts and evident manipulation, the system (9) is reduced to the form

$$u = b_1 w_{,r} + C_1 r + C_2/r; \;\; \psi = b_2 w_{,r} + C_3 r + C_4/r; \;\; L_3(w_{,r}) + M^4 \ddot{w} = 0, \tag{12}$$

where

$$\begin{gathered} b_1 = (a_3 a_4 - a_2 a_5)/(a_1 a_4 - a_2^2); \;\; b_2 = (a_1 a_5 - a_2 a_3)/(a_1 a_4 - a_2^2); \\ M^4 = \frac{M_0 r_0^2 a_1 (a_1 a_4 - a_2^2)}{(a_1 a_6 - a_3^2)(a_1 a_4 - a_2^2) - (a_1 a_5 - a_2 a_3)^2}. \end{gathered}$$

Because of the continuity of the desired solution at the origin for solid plates, it is necessary to set $C_2 = C_4 = 0$. There follows from the condition (11) on the boundary

$$C_1 = -b_1 w_{,r}; \;\; C_3 = -b_2 w_{,r} \;\; (r = 1),$$

which permits obtaining a boundary condition for the deflection

$$\begin{gathered} w = 0; \;\; a_7 w_{,rr} + a_8 w_{,r} + M_t = 0 \;\; (r = 1); \\ a_7 = a_{60} - b_1 a_2 - b_2 a_5; \;\; a_8 = a_{60} + b_1 a_2 + b_2 a_5. \end{gathered} \tag{13}$$

We represent the solution of (12) in the form of the sum of quasistatic deflection and a dynamic part

$$w = w_s + w_d. \tag{14}$$

The quasistatic deflection satisfies the equation

$$L_3(w_{s,r}) = 0$$

for the boundary conditions

$$w_s = 0;\ a_7 w_{s,rr} + a_8 w_{,r} + M_t = 0\ (r = 1).$$

For a solid plate

$$w_s = \frac{M_t(1-r^2)}{2(a_7+a_8)}. \tag{15}$$

Substituting the solution (14) into (11) and the conditions (10) and (13) and taking account of the expression (15), we obtain an equation to determine the dynamic part of the deflection

$$L_3(w_{d,r}) + M^4\ddot{w}_d = -\frac{M^4(1-r^2)}{2(a_7+a_8)}\ddot{M}_t \tag{16}$$

under the initial and boundary conditions

$$w_d = -\frac{1-r^2}{2(a_7+a_8)}M_t;\quad \dot{w}_d = -\frac{1-r^2}{2(a_7+a_8)}\dot{M}_t\ (t=0);$$
$$w_d = 0;\ a_7 w_{d,rr} + a_8 w_{d,r} = 0\ (r=1). \tag{17}$$

Let us first examine the homogeneous differential equation corresponding to (16). Its solution is assumed in the form

$$w_d^0 = v(r)(A\cos\omega t + B\sin\omega t). \tag{18}$$

After substituting (18) in this homogeneous equation, we obtain a differential equation to determine the function v(r):

$$L_3(v_{,r}) - \beta^4 v = 0;\ \beta^4 = \omega^2 M^4. \tag{19}$$

Taking account of boundedness at the origin, the solution of (19) is the following

$$v = C_5 J_0(\beta r) + C_6 I_0(\beta r). \tag{20}$$

Here J_0, I_0 are zero-order Bessel function of the first kind of real and imaginary arguments. Substituting (20) into the boundary conditions (17) and requiring the solution of the obtained system of equations in the constants of integration C_1, C_2 not be trivial, we obtain a transcendental equation to determine the eigennumbers β_n:

$$J_0(\beta)[a_7(\beta I_0(\beta) - I_1(\beta)) + a_8 I_1(\beta)] + I_0(\beta)[a_7(\beta J_0(\beta) - J_1(\beta)) + a_8 J_1(\beta)]. \tag{21}$$

The natural vibrations frequencies are afterwards determined from the formula

$$\omega_n = \beta_n^2/M^2. \tag{22}$$

An orthonormalized system of eigenfunctions $v_n \equiv v(\beta_n r)$

$$v_n = \frac{1}{d_n}\left[J_0(\beta_n r) - \frac{J_0(\beta_n)}{I_0(\beta_n)} I_0(\beta_n r)\right] \tag{23}$$

is introduced to describe the dynamic part of the deflection of the system under investigation. The normalizing factor d_n is determined from the requirement of orthonormality of the system v_n

$$d_n^2 = \int_0^1 \left[J_0(\beta_n r) - \frac{J_0(\beta_n)}{I_0(\beta_n)} I_0(\beta_n r)\right]^2 r\,dr = \frac{1}{2}[J_1^2(\beta_n) + I_1^2(\beta_n)] + \frac{J_0(\beta_n)}{\beta_n}\left[J_1(\beta_n) + \frac{I_1(\beta_n)}{I_0(\beta_n)} J_0(\beta_n)\right].$$

The desired deflection w_d that satisfies the inhomogeneous equation (16) is represented by using a series expansion in a functional system of functions (23):

$$w_d = \sum_{n=0}^{\infty} v_n q_n(t). \quad (24)$$

Substituting this series into (16) and the conditions (17) and taking (19) into account, multiplying the terms of the equation by $v_n rdr$ and integrating with respect to r between zero and one, we obtain an equation for q_n

$$\ddot{q}_n + \omega_n^2 q_n = -\frac{I(\beta_n)}{2(a_7 + a_8)} \ddot{M}_t \quad (25)$$

under the conditions

$$q_n = -\frac{I(\beta_n)}{2(a_7 + a_8)} M_t(0); \; \dot{q}_n = -\frac{I(\beta_n)}{2(a_7 + a_8)} \dot{M}_t(0) \; (t = 0). \quad (26)$$

Here

$$I(\beta_n) = \int_0^1 (1 - r^2) v_n rdr = \frac{1}{d_n \beta_n^4} \left[\beta_n^2 \left(J_1(\beta_n) - \frac{J_0(\beta_n)}{I_0(\beta_n)} I_1(\beta_n) \right) + 2J_0(\beta_n) S_{2,-1}(\beta_n) + J_1(\beta_n) S_{3,0}(\beta_n) \right]; \quad (27)$$

when $S_{\mu,\nu}$ are Lommel functions.

The solution of the equation can be written down in the form

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t - \frac{I(\beta_n)}{2(a_7 + a_8)\omega_n} \int_0^t \sin[\omega_n (t - \tau)] \ddot{M}_t(\tau) d\tau. \quad (28)$$

The constants of integration A_n, B_n are determined from the initial conditions (26):

$$A_n = -\frac{I(\beta_n)}{2(a_7 + a_8)} M_t(0); \; B_n = -\frac{I(\beta_n)}{2(a_7 + a_8)\omega_n} \dot{M}_t(0). \quad (29)$$

The quasistatic deflection (15) can be expanded in a series of eigenfunctions v_n:

$$w_s = \frac{M_t}{2(a_7 + a_8)} \sum_{n=0}^{\infty} I(\beta_n) v_n, \quad (30)$$

where $I(\beta_n)$ is defined by (27).

We obtain the total dynamic deflection of the plate by summing (24) and (30):

$$w = \sum_{n=0}^{\infty} v_n \left(q_n + \frac{M_t}{2(a_7 + a_8)} I(\beta_n) \right). \quad (31)$$

The radial displacements and shear follow from the relationships (12):

$$u = b_1 \sum_{n=0}^{\infty} v_{n,r} \left(q_n + \frac{M_t I(\beta_n)}{2(a_7 + a_8)} \right) + c_1 r;$$

TABLE 1. Eigennumbers of Equation (21)

n	β_n	n	β_n
0	2,525	10	18,095
1	2,655	11	18,115
2	5,585	12	21,225
3	5,645	13	21,245
4	8,695	14	24,365
5	8,735	15	24,385
6	11,825	16	27,505
7	11,855	17	27,515
8	14,955	18	30,645
9	14,975	19	30,655

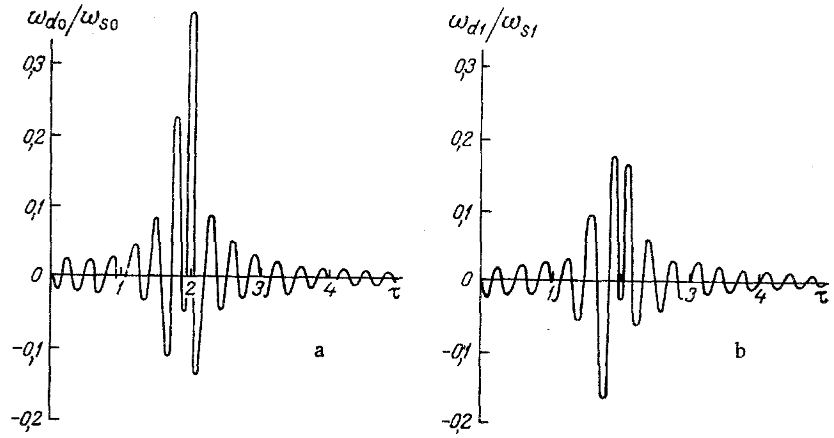

Fig. 1. Dependence of the ratio between the dynamic component of the deflection w_{dn} and the quasistatic component w_{sn} on the dimensionless time $\tau = (a_{1t}t)/h^2$ for fundamental tone vibrations frequencies: a) n = 0 ($\omega_0 r_0$ = 585.3 m/sec); b) n = 1 ($\omega_1 r_0$ = 647.1 m/sec).

$$\psi = b_2 \sum_{n=0}^{\infty} v_{n,r} \left(q_n + \frac{M_t I(\beta_n)}{2(a_7 + a_8)} \right) + c_3 r. \qquad (32)$$

Here q_n is determined by (28) with (29) taken into account:

$$v_{n,r} = -\frac{\beta_n}{d_n}\left[J_1(\beta_n r) + \frac{J_0(\beta_n)}{I_0(\beta_n)} I_1(\beta_n r) \right];$$

$$\{c_1,\ c_3\} = \{b_1,\ b_2\} \sum_{n=0}^{\infty} \frac{\beta_n}{d_n}\left[J_1(\beta_n) + \frac{J_0(\beta_n)}{I_0(\beta_n)} I_1(\beta_n) \right]\left[q_n + \frac{M_t I(\beta_n)}{2(a_7 + a_8)} \right].$$

Therefore, the functions describing the vibrations of a sandwich plate under thermal impact are determined by (31) and (32).

Numerical results were obtained on an ES-1022 electronic computer for a sandwich plate whose heat shielding layer is from cordierite, the filler from a fluoroplastic, and the carrying layers from D16T aluminum alloy. The thermophysical and elastic characteristics of the materials mentioned were determined from known experimental data [7-9].

The transcendental equation (21) for the eigennumbers was investigated in a 0-50 interval. Thirty-two roots were obtained. The first 20, calculated to 0.005 accuracy, are presented in Table 1. They correspond to the layer geometric parameters h_3 = 10, h_1 = 20, h_2 = 0.05 and grown as the total layer thicknesses grow. Their corresponding frequencies $\omega_n r_0$ (m/sec) vary between the limits 585-86,270. The ratio between the deflection dynamic (24) and quasistatic components for the first two frequencies are represented in the figure as a function of the dimensionless time. The extremal splashes of the ratio corresponding to the functional tone frequency $\omega_0 r_0$ are compensated partially in the region τ = 2 upon summing the series (31) by means of its dual frequency $\omega_1 r_0$. For τ > 5 the amplitude of the vibrations diminishes gradually.

Investigation of the temperature field im the inner carrying layer showed that it can be assumed constant for relatively thin layers $h_2 \ll h_3$.

NOTATION

r, φ, z, cylindrical coordinate system; q_t, heat flux; T, temperature, t, time; λ_k, c_{kt}, a_{kt}, heat conduction, specific heat, and thermal diffusivity of the k-th layer; ρ_k, density; k, layer number; h_k, layer thickness; r_0, plate radius; u_r^k, radial displacements in the layers; u, radial displacements of the filler middle plane; ψ, shear in the filler; w, plate deflection; w_d, w_s, dynamic and quasistatic deflection components; G_k, K_k, shear and bulk deformation moduli; σ_α^k, ε_α^k, stress and strain tensor components; σ_α^k, ε_α^k, global parts of the stress and strain tensors; Π, strain potential energy; M_r, radial generalized moment;

M_t, temperature component of the generalized moment; L_2, L_3, linear differential operators; β_n, eigennumbers; v_n, orthonormalized eigenfunctions; and ω_n, natural vibrations frequencies.

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